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The Ising model with long-range ferromagnetic interactions

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Abstract. A variational method based on the nearest-neighbour Ising model is used to derive an expression for the low-temperature correlation function in the Ising model with long-range ferromagnetic interactions. Also by expanding the correlation function directly, we demonstrate that the characteristic power-law behaviour at large distances is irrelevant in the critical region, when above the marginal range of interaction.

1. Introduction

Ruelle (1968) proved that for the one-dimensional Ising model with Hamiltonian

$$H = -\sum_{i,n}^{\infty} J(n)\sigma_i\sigma_{i+n} \tag{1}$$

where $J(n) \geqslant 0$, $n = 1, 2, 3, \ldots$, and $M_0 = \Sigma_{n=1}^{\infty} J(n)$ is finite, no phase transition exists provided that $M_1 = \Sigma_{n=1}^{\infty} nJ(n)$ is finite. For power-law potentials $J(n) = J/n^{1+\sigma}$, this condition implies the absence of long-range order at all finite temperatures for $\sigma > 1$, i.e. for interactions decaying faster than inverse square.

Kac and Thompson (1969) conjectured that a phase transition does exist for $0 < \sigma \leqslant 1$, but Dyson (1969) was only able to prove it for $0 < \sigma < 1$, leaving the borderline case $\sigma = 1$ undecided. Cardy (1984) found a transition to occur for $J(n) = J/n^2$, where the correlation length diverges according to $\ln(\xi/\xi_0) \sim t^{-1/2}$, where $t \equiv T/T_c - 1$, which is a strong singularity of the Kosterlitz-Thouless type (Kosterlitz 1974).

Fisher *et al* (1972) investigated the d-dimensional model with isotropic potential $J(r) \sim J/r^{d+\sigma}$. Using the momentum space RG, they analysed the stability of the long-range fixed point, and found that the following 'classical' or Gaussian exponents were exact for $\sigma < d/2$:

$$\eta = 2 - \sigma \qquad \nu = 1/\sigma \qquad \gamma = 1. \tag{2}$$

On the long-range Gaussian border, $\sigma = d/2$, the classical exponents in (2) have logarithmic corrections, and for $d/2 < \sigma < 2$, Fisher *et al* performed expansions for the exponents in powers of $\varepsilon' = 2\sigma - d$ for fixed $\sigma \neq 2$ and in $\Delta\sigma = \sigma - d/2$ for fixed d. However the prediction of $\sigma = 2$ as the margin of short-range behaviour is inconsistent with what is known about the long-range Ising chain, and for $d \leqslant 3$ leads to an unexpected jump in the value of η across the margin.

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Sak (1973) showed that the above problems could be resolved if the universality crossover occurs at $\sigma = 2 - \eta_{SR}$ and not at $\sigma = 2$. The discontinuity in η is removed, since $\eta_{LR} = 2 - \sigma = \eta_{SR}$ at $\sigma = 2 - \eta_{SR}$. Moreover, in $d = 1$ where $\eta_{SR} = 1$, a crossover is now correctly predicted at $\sigma = 1$. Subsequently Bray (1986), in the context of the random field model with long-range exchange and/or long-range correlated random fields, has rederived the result $\sigma_m = 2 - \eta_{SR}$ for the pure long-range system, where we denote the marginal range by σ_m. Also, van Enter (1982) found the long range contribution to the energy in the XY model for $d \geqslant 3$ to be unimportant for $\sigma > \sigma_m$.

We will show that the variational method based on the short-range Ising model (Takahashi 1981) can also predict the marginal interaction range. In addition, the low-temperature correlation function will be found to have a nearest-neighbour form with a modified correlation length. By performing a direct perturbation expansion of the nearest-neighbour correlation function, we will demonstrate that the long-range power-law part is irrelevant for potentials in the short-range universality class.

The order of the paper is as follows. The free energy of the Ising chain is first expanded, and from it the low-temperature correlation function is derived by taking the derivative with respect to the long-range potential. The method is then generalised to the d-dimensional long-range model. We perform a direct expansion of the correlation function of the Ising chain, which enables us to demonstrate the existence of power-law correlations at large distances. These do not appear to have been studied extensively in magnets, but are nevertheless interesting features of the long-range model, and are relevant to the discussion of critical properties. The correlation function is expanded in d dimensions and used to examine the long-range contribution to the critical susceptibility, in analogy with work done by others on classical liquids.

2. Variational method for the long-range Ising chain

Following Takahashi (1981), we begin by writing the Hamiltonian of the long-range Ising chain in (1) as

$$H = H_0 + (H - H_0) \tag{3}$$

where

$$H_0 = -\mathscr{J}_0 \sum_i^N \sigma_i \sigma_{i+1} \tag{4}$$

is the exactly soluble nearest-neighbour Hamiltonian. The variational inequality can be applied in order to replace the true free energy F by the minimum with respect to $\mathscr{J}_0$ of $F_0 + \langle H - H_0 \rangle_0$. Doing this leads to the following expression for $f \equiv F/N$ as $N \to \infty$:

$$f(\beta, \mathscr{J}_0) = f_0 - \sum_{n=1}^{\infty} J(n) \tanh^n(\beta \mathscr{J}_0) + \mathscr{J}_0 \tanh \beta \mathscr{J}_0 \tag{5}$$

where

$$f_0 = -\beta^{-1} \ln(2 \cosh \beta \mathscr{J}_0). \tag{6}$$

$\mathscr{J}_0$ is given by

$$(1 - \tanh^2(\beta \mathscr{J}_0)) \left(\mathscr{J}_0 - \sum_{n=1}^{\infty} n J(n) \tanh^{n-1}(\beta \mathscr{J}_0) \right) = 0 \tag{7}$$

and the correlation length ξ_0 has the nearest-neighbour form

$$\xi_0(\mathcal{J}_0) = \frac{-1}{\ln \tanh(\beta \mathcal{J}_0)}. \tag{8}$$

The effective long-range correlation function is given by the nearest-neighbour correlation function

$$C(n) = C_0(n) = \exp(-n/\xi_0). \tag{9}$$

The effects of $J(n)$ are therefore felt through $\mathcal{J}_0$ and hence through ξ_0. This complies with the standard definition of a correlation function

$$C(n) = -\frac{\partial f}{\partial J(n)}. \tag{10}$$

Using our variational free energy (5) in equation (10), we obtain $C(n) = C_0(n)$.

An Ising model with power-law ferromagnetic interactions displays power-law correlations at sufficiently large distances. It is an important point, therefore, that our approximation gives only exponential correlations, although it is likely that these will be valid at low enough temperatures, since the model can then be mapped onto a kink or domain-wall mean-field theory, analogous to that of a nearest-neighbour chain. Just as $2J$ is the wall energy dividing two opposite domains in the nearest-neighbour chain, $2\mathcal{J}_0$ is the wall energy for the long-range model. We can express the definition of $\mathcal{J}_0$ (finite) from equation (7) as

$$2\mathcal{J}_0 = 2J \exp(1/\xi_0) \sum_{n=1}^{\infty} nJ(n)C(n) \tag{11}$$

which is indeed the excitation energy of an isolated kink when $\xi_0 \gg 1$.

There is a phase transition for $J(n) = J/n^2$ but not for $J(n) = J/n^{1+\sigma}$, $\sigma > 1$. A test of the variational approximation is whether it can confirm this. Considering the expression for finite $\mathcal{J}_0$ with $J(n) = J/n^{1+\sigma}$:

$$\mathcal{J}_0/J = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \tanh^{n-1}(\beta \mathcal{J}_0) \tag{12}$$

it can be seen that for $\sigma > 1$ there is indeed no singularity in $\mathcal{J}_0$ and therefore none in ξ_0 for $T > 0$. This is because for $T > 0$ ($\beta < \infty$) an upper bound on $\mathcal{J}_0$ is given by

$$\mathcal{J}_0/J = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \qquad \sigma > 1 \tag{13}$$

which is finite.

We consider the case $\sigma = 1$ explicitly in order to find the transition temperature predicted from this method; we follow a procedure suggested by the referee.

The free energy (equation (5)) is given below for $\sigma = 1$:

$$f(\beta, \mathcal{J}_0) = -\beta^{-1} \ln 2 \cosh(\beta \mathcal{J}_0) - J \sum_{n=1}^{\infty} \frac{1}{n^2} \tanh^n(\beta \mathcal{J}_0) + \mathcal{J}_0 \tanh(\beta \mathcal{J}_0). \tag{14}$$

By expanding $f(\beta, \mathcal{J}_0)$ in powers of $\exp(-2\beta \mathcal{J}_0)$, for large $\beta \mathcal{J}_0$ (see the appendix), we find:

$$\lim_{\mathcal{J}_0 \to \infty} f(\beta, \mathcal{J}_0) = -J\zeta(2) + (a\mathcal{J}_0 + b) \exp(-2\beta \mathcal{J}_0) \tag{15}$$

where $a=2(2\beta J-1)$, $b=\beta^{-1}(2\beta J-1-2\beta J\ln 2)$, and $\zeta(2)\simeq 1.645$ is the value of the Riemann zeta function, so that $-J\zeta(2)$ is the ground-state free energy density.

The free energy has a minimum for $\mathscr{J}_0\to\infty$ provided that $a>0$. For all $a\leqslant 0$, the minimum occurs for $\mathscr{J}_0$ *finite*, and there is thus a *first*-order transition. This method predicts $2\beta_c J=1$, or $kT_c=2J$, which compares more favourably with the result found by Nagle and Bonner (1970) of $kT_c=1.6J$, than the mean-field theory result of $kT_c\simeq 3.2J$. (The numerical solution of these equations found by Takahashi (1981) failed to locate the first-order transition and gave $kT_c\simeq 2.4J$.) It is interesting that this method which is based on the nearest-neighbour chain (which does not order) does lead to a prediction for T_c at $\sigma=1$. It does not, however, agree with the exponents found by Cardy (1984), who used a scaling analysis.

3. The variational method in d dimensions

The variational free energy density in d dimensions can be expressed as

$$f=f_0-A(d)\int_1^\infty J(r)C_0(r)r^{d-1}\,\mathrm{d}r+\mathscr{J}_0C_0(1) \tag{16}$$

where f_0 is the nearest-neighbour free energy density in d dimensions, $J(r)$ is the isotropic interaction and $C_0(r)$ the nearest-neighbour correlation function. The factor r^{d-1} in the integrand relates to the d-dimensional surface area and $A(d)$ represents the angular contribution to the spherical integral in d dimensions.

Putting $J(r)=J/r^{d+\sigma}$ for $\sigma\geqslant 2-\eta_{SR}$, the integral over $J(r)$ becomes, near the critical point,

$$\text{constant}+J\int_a^\infty \frac{g(r/\xi_0)}{r^{d+1+\sigma-2+\eta_{SR}}}\,\mathrm{d}r \tag{17}$$

where the constant term depends only on the lattice spacing and the cutoff a, and the critical correlation function $C_0(r)=g(r/\xi_0)r^{-(d-2+\eta_{SR})}$, where $g(r/\xi_0)$ is the scaling function. The second term in (17) gives a contribution of $O(\xi_0^{-d}\xi_0^{-(\sigma-\sigma_m)})$. Hence the contribution of the long-range Hamiltonian $\langle H_{LR}\rangle_0$, to the variational free energy density is negligible in comparison with the short-range free energy density when $\sigma>\sigma_m=2-\eta_{SR}$, but dominates for $\sigma<2-\eta_{SR}$, in agreement with Sak (1973), van Enter (1982) and Bray (1986).

4. Expansion of the correlation function in the long-range chain

Asymptotic power-law correlations have been proved to exist in the Ising model by Griffiths (1967a), in the Ising chain for all $\sigma>0$ at high temperatures by Naimnhanov (1979), and by Benfatto (1984) in continuous systems at all temperatures. The key question arising from the fact that power-law correlations exist is the following: 'How can this be reconciled with the analysis of Cardy (1984) which states that potentials $J(n)\sim 1/n^{1+\sigma}$, for $\sigma>1$, exhibit *short-range* critical behaviour, whereas we know correlations are only *exponential* in the nearest-neighbour model?' We will show that power-law correlations associated with potentials in the short-range universality class are irrelevant in the critical region, near $T_c=0$.

We can expand the correlation function for the Ising chain

$$C(n) \equiv \langle \sigma_0 \sigma_n \rangle = \frac{\mathrm{Tr}\, \sigma_0 \sigma_n \exp[-\beta(H_0 + H_1)]}{\mathrm{Tr} \exp[-\beta(H_0 + H_1)]} \tag{18}$$

in powers of βH_1, where

$$H_0 = -J_0 \sum_i \sigma_i \sigma_{i+1} \qquad H_1 = -\sum_{\substack{i=1\\ j=m}} J_1(n) \sigma_i \sigma_{i+j} \tag{19}$$

and m is the arbitrary short-range cut-off of the long-range potential. The result is

$$\begin{aligned} C(n) &= \langle \sigma_0 \sigma_n \rangle_0 \\ &\quad + \langle \beta H_1 \rangle_0 \langle \sigma_0 \sigma_n \rangle_0 - \langle \sigma_0 \sigma_n \beta H_1 \rangle_0 \\ &\quad + \tfrac{1}{2}(\langle \sigma_0 \sigma_n (\beta H_1)^2 \rangle_0 + \langle \sigma_0 \sigma_n \rangle_0 \langle \beta H_1 \rangle_0^2 - \langle \sigma_0 \sigma_n \rangle_0 \langle (\beta H_1)^2 \rangle_0 - \langle \sigma_0 \sigma_n \beta H_1 \rangle_0 \langle \beta H_1 \rangle_0) \\ &\quad + \ldots . \end{aligned} \tag{20}$$

To $O(\beta H_1)$ only,

$$C(n) = C_0(n) + \beta \sum_{i,j} J_1(j)(\langle \sigma_0 \sigma_n \sigma_i \sigma_{i+j} \rangle_0 - \langle \sigma_0 \sigma_n \rangle_0 \langle \sigma_i \sigma_{i+j} \rangle_0) \tag{21}$$

where $C_0(n) \equiv \langle \sigma_0 \sigma_n \rangle_0 = \exp(-n/\xi_0)$ and $\xi_0 = -1/\ln[\tanh(\beta J_0)]$. We wish to investigate the power-law tails which show up at large distances, $n/\xi_0 \gg 1$, where $C_0(n)$ can be neglected. Now we can split terms of the form $\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle$ into products of pairwise averages $\langle \sigma_1 \sigma_2 \rangle \langle \sigma_3 \sigma_4 \rangle$, $\langle \sigma_1 \sigma_3 \rangle \langle \sigma_2 \sigma_4 \rangle$ etc, by invoking the Griffiths inequality (Griffiths 1967b)

$$\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle \geqslant \langle \sigma_i \sigma_j \rangle \langle \sigma_k \sigma_l \rangle . \tag{22}$$

Doing this for $\langle \sigma_0 \sigma_n \sigma_i \sigma_{i+j} \rangle_0$ in (21) and taking $J_1(j) = J_1/j^{1+\sigma}$ gives

$$C(n) \geqslant \beta J_1 \sum_{i,j} \frac{\langle \sigma_0 \sigma_i \rangle_0 \langle \sigma_n \sigma_{i+j} \rangle_0}{j^{1+\sigma}} . \tag{23}$$

To proceed, we shall approximate summations over the 1D lattice by integrals,

$$\begin{aligned} C(n) &\simeq \beta J_1 \int_1^\infty \mathrm{d}x \int_1^\infty \mathrm{d}y \frac{C_0(x) C_0(x+y-n)}{y^{1+\sigma}} \\ &= \beta J_1 \int_1^\infty \frac{\mathrm{d}y}{y^{1+\sigma}} \int_1^\infty \mathrm{d}x \exp(-x/\xi_0) \exp(-|x+y-n|/\xi_0). \end{aligned} \tag{24}$$

The integrations are dominated by the regions within a range ξ_0 of the spins σ_0 and σ_n, giving

$$C(n) \simeq \frac{4 \beta J_1 \xi_0^2}{n^{1+\sigma}} \qquad \text{for } n/\xi_0 \gg 1. \tag{25}$$

The result of expanding to $O(\beta H_1)$ is that the correlation function comprises a nearest-neighbour part and a term which decays like $J_1(n)$ at distances large compared with the nearest-neighbour correlation length. We can compare this result, derived on the physical basis of correlated regions of spins interacting over large distances through the long-range potential $J_1(n)$, with the formal expansion (following Brout 1965)

$$\begin{aligned} C(n) &= C_0(n) + \beta J_1(n)(\chi_0/\beta)^2 \\ &\quad + \sum_{n=1}^{\infty} \beta^{-2} \sum_{i_1 \ldots i_n} J_1(i_1) J_1(|i_2 - i_1|) \ldots J_1(|n - i_n|) \chi_0^{n+2} \end{aligned} \tag{26}$$

where χ_0 is the nearest-neighbour zero-field susceptibility. Our approximate expression in (25) is qualitatively consistent with the first two terms of (26), since at high temperatures the susceptibility $\chi_0 \simeq \beta\xi_0$.

The significance of the long-range correlations depends on their contribution to thermodynamic functions. Let us apply the definition of susceptibility

$$\chi = \beta \int_1^\infty C(n)\,\mathrm{d}n \tag{27}$$

to find the low-temperature (critical) susceptibility as a function of the diverging correlation length ξ_0. This leads to

$$\chi(\xi_0) = \chi_0(\xi_0) + (4\beta^2 J_1)\xi_0^2 \int_{p\xi_0}^\infty \frac{\mathrm{d}n}{n^{1+\sigma}} \tag{28}$$

where $p \gg 1$ is a parameter determining the short-distance cut-off of the long-range contribution to the correlation function. There is no real physical cut-off at $n = p\xi_0$, but this distance does mark the limit of our derivation. It can easily be seen, however, that the unknown long-range correlation function up to $n = p\xi_0$ would not make a contribution to the susceptibility of more than the order of ξ_0. Performing the integral in (28) leads to

$$\chi(\xi_0) \simeq \beta\xi_0 + \left(\frac{4\beta^2 J_1}{\sigma p^\sigma}\right)\xi_0^{2-\sigma} \tag{29}$$

where we have substituted

$$\chi_0(\xi_0) \equiv \beta \int_1^\infty \exp(-n/\xi_0)\,\mathrm{d}n \simeq \beta\xi_0. \tag{30}$$

Formally, we can write $\chi(\xi_0) = \chi_{\mathrm{SR}}(\xi_0) + \chi_{\mathrm{LR}}(\xi_0)$, where the ratio of long-range to short-range contributions to the critical susceptibility is given by

$$\frac{\chi_{\mathrm{LR}}(\xi_0)}{\chi_{\mathrm{SR}}(\xi_0)} = \left(\frac{4\beta J_1}{\sigma p^\sigma}\right)\xi_0^{1-\sigma}. \tag{31}$$

For $\sigma > 1$, $\chi_{\mathrm{LR}}(\xi_0)$ is irrelevant in comparison to $\chi_{\mathrm{SR}}(\xi_0)$, and therefore $\chi \sim \xi_0$, in accord with short-range scaling.

5. Correlation function expansion in *d* dimensions

Following the approach in one dimension, it is possible to make an extension to any d, as we did for the variational approximation.

In one dimension the method was based upon spin clusters of length ξ_0 interacting through a $1/n^{1+\sigma}$ potential at distances large compared with ξ_0. Similarly in general dimension d, we can take clusters of volume $O(\xi_0^d)$ to be interacting through a $1/r^{d+\sigma}$ potential at distances large compared with ξ_0. The analogue of (24) is

$$C(r) - C_0(r) = A(d)\beta J_1 \left(\int_1^{\xi_0} C_0(r) r^{d-1}\,\mathrm{d}r\right)^2 r - (d+\sigma) \tag{32}$$

where

$$\int_1^{\xi_0} C_0(r) r^{d-1}\,\mathrm{d}r \simeq \int_1^a r^{d-1}\,\mathrm{d}r + \int_a^{\xi_0} \frac{\exp(-r/\xi_0) r^{d-1}}{r^{d-2+\eta_{\mathrm{SR}}}}\,\mathrm{d}r \tag{33}$$

using our previous notation. As $\xi_0 \to \infty$, the first term in (33) of $O(a^d)$ can be neglected in comparison with the second term, which is of order $\xi_0^{2-\eta_{SR}}$. The third term is small, and so can also be neglected. Therefore from the d-dimensional susceptibility which we define as

$$\chi = \beta \int_1^\infty C(r) r^{d-1}\, dr \tag{34}$$

we find that

$$\chi = \chi_0(\xi_0) + O\left((\xi_0^{2-\eta})^2 \int_{p\xi_0}^\infty \frac{dr}{r^{1+\sigma}}\right)$$
$$\sim \xi_0^{2-\eta} + \xi_0^{2-\eta}/\xi_0^{\sigma-(2-\eta)}. \tag{35}$$

Hence the ratio of long-range to short-range contributions to the d-dimensional critical susceptibility is given by

$$\frac{\chi_{LR}(\xi_0)}{\chi_{SR}(\xi_0)} \sim \frac{1}{\xi_0^{\sigma-\sigma_m}} \tag{36}$$

where $\sigma_m = 2 - \eta_{SR}$ defines the marginal range. As we expect, the long-range contribution to the critical susceptibility is irrelevant for $\sigma > \sigma_m$ and marginal at $\sigma = \sigma_m$. Our reasoning is the magnetic analogue of that of Kayser and Raveché (1984), who investigated the influence on critical scaling of the weak but long-range ($1/r^6$) intermolecular interactions in neutral fluids.

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Appendix

The expansion of (14) in powers of $\exp(-2\beta\mathcal{J}_0)$ is performed for the three terms separately; the first and third terms are trivial:

$$-\beta \ln 2\cosh(\beta\mathcal{J}_0) + \mathcal{J}_0 \tanh(\beta\mathcal{J}_0)$$
$$\simeq -\mathcal{J}_0 - \beta^{-1}\exp(-2\beta\mathcal{J}_0) + \mathcal{J}_0[1 - 2\exp(-2\beta\mathcal{J}_0)] + O(\exp(-4\beta\mathcal{J}_0)). \tag{A1}$$

The second term can be re-expressed as an integral:

$$-J\sum_{n=1}^\infty \frac{1}{n^2}\tanh^n(\beta\mathcal{J}_0)$$
$$= J\int_0^{\tanh(\beta\mathcal{J}_0)} dx \frac{1}{x}\ln(1-x)$$
$$= J\int_0^1 dx\frac{1}{x}\ln(1-x) - J\int_{1-\tanh(\beta\mathcal{J}_0)}^1 dx\frac{1}{x}\ln(1-x)$$
$$= J\zeta(2) - J\int_0^{2\exp(-2\beta\mathcal{J}_0)} dy\frac{1}{1-y}\ln y$$
$$= J\zeta(2) - 2J\exp[(-2\beta\mathcal{J}_0)(\ln 2 - 2\beta\mathcal{J}_0 - 1)] + O(4\exp(-4\beta\mathcal{J}_0)). \tag{A2}$$

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